

# CHARACTERIZATIONS OF THE QUATERNIONIC BERTRAND CURVE IN EUCLIDEAN SPACE $E^4$

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**ABSTRACT.** In [18], *L. R. Pears* proved that Bertrand curves in  $\mathbb{E}^n$  ( $n > 3$ ) are degenerate curves. This result restate in [16] by *Matsuda and Yorozu*. They proved that *there is no special Bertrand curves in  $E^n$  ( $n > 3$ )* and they define new kind of Bertrand curves called  $(1, 3)$ -type Bertrand curves in 4-dimensional Euclidean space.

In this study, we define a **quaternionic Bertrand curve**  $\alpha^{(4)}$  in Euclidean space  $E^4$  and investigate its properties for two cases. In the first case; we consider *quaternionic Bertrand curve in the Euclidean space  $E^4$*  for  $r - K = 0$  where  $r$  is the torsion of the spatial quaternionic curve  $\alpha$ ,  $K$  is the principal curvature of the quaternionic curve  $\alpha^{(4)}$ . And then, in the other case, we prove that *there is no quaternionic Bertrand curve in the Euclidean space  $E^4$*  for  $r - K \neq 0$ . So, we give an idea of quaternionic Bertrand curve which we call **quaternionic  $(N - B_2)$  Bertrand curve in the Euclidean space  $E^4$**  by using the similar method in [16] and we give some characterizations of such curves.

## 1. INTRODUCTION

The general theory of curves in a Euclidean space ( or more generally in a Riemannian Manifolds) have been developed a long time ago and we have a deep knowledge of its local geometry as well as its global geometry. In the theory of curves in Euclidean space, one of the important and interesting problem is characterizations of a regular curve. In the solution of the problem, the curvature functions  $k_1$  (or  $\kappa$ ) and  $k_2$  (or  $\tau$ ) of a regular curve have an effective role. For example: if  $k_1 = 0 = k_2$ , then the curve is a geodesic or if  $k_1 = \text{constant} \neq 0$  and  $k_2 = 0$ , then the curve is a circle with radius  $(1/k_1)$ , etc. Thus we can determine the shape and size of a regular curve by using its curvatures. Another way in the solution of the problem is the relationship between the Frenet vectors of the curves (see [13]). For instance Bertrand curves:

In 1845, *Saint Venant*(see [19]) proposed the question whether upon the surface generated by the principal normal of a curve, a second curve can exist which has for its principal normal the principal normal of the given curve. This question was answered by *Bertrand* in 1850 in a paper(see [5]) in which he showed that a necessary and sufficient condition for the existence of such a second curve is that a linear relationship with constant coefficients shall exist between the first and second curvatures of the given original curve. In other word, if we denote first and second curvatures of a given curve by  $k_1$  and  $k_2$  respectively, then for  $\lambda, \mu \in \mathbb{R}$  we have  $\lambda k_1 + \mu k_2 = 1$ . Since the time of Bertrand's paper, pairs of curves of this kind have

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been called *Conjugate Bertrand Curves*, or more commonly *Bertrand Curves* (see [13]).

In 1888, *C. Bioche* give a new theorem in [6] to obtaining Bertrand curves by using the given two curves  $C_1$  and  $C_2$  in Euclidean 3-space. Later, in 1960, *J. F. Burke* in [10] give a theorem related with *Bioche's theorem on Bertrand curves*.

The following properties of Bertrand curves are well known: If two curves have the same principal normals, **(i)** corresponding points are a fixed distance apart; **(ii)** the tangents at corresponding points are at a fixed angle. These well known properties of Bertrand curves in Euclidean 3-space was extended by *L. R. Pears* in [18] to Riemannian  $n$ -space and found general results for Bertrand curves. When we apply these general results to Euclidean  $n$ -space, it is easily found that either  $k_2$  or  $k_3$  is zero; in other words, Bertrand curves in  $\mathbb{E}^n$  ( $n > 3$ ) are degenerate curves. This result restate in [16] by *Matsuda and Yorozi*. They proved that *there is no special Bertrand curves in  $E^n$  ( $n > 3$ )* and they define new kind, which is called  $(1, 3)$ -type Bertrand curves in 4-dimensional Euclidean space. Bertrand curves and their characterizations were studied by many authours in Euclidean space as well as in Riemann–Otsuki space, in Minkowski 3-space and Minkowski spacetime (for instance see [3, 4, 7, 12, 15, 20, 21]).

The quaternion was introduced by Hamilton. His initial attempt to generalize the complex numbers by introducing a three-dimensional object failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. On the 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar(real) axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vectors axes.

In 1987, The Serret-Frenet formulas for a quaternionic curve in  $E^3$  and  $E^4$  was defined by Bharathi and Nagaraj [11] and then in 2004, Serret-Frenet formulas for quaternionic curves and quaternionic inclined curves have been defined in Semi-Euclidean space by Çöken and Tuna in 2004 [1].

In 2011 Gök et.al [8, 9] defined a new kind of slant helix in Euclidean space  $E^4$  and semi-Euclidean space  $E_2^4$ . It called quaternionic  $B_2$ -slant helix in Euclidean space  $E^4$  and semi-real quaternionic  $B_2$ -slant helix in semi-Euclidean space  $E_2^4$ , respectively. In 2011 Gungor and Tosun gived some characterizations of quaternionic rectifying curve.

In this study, we define a quaternionic Bertrand curve  $\alpha^{(4)}$  and investigate its properties for two cases. In the first case; we consider *quaternionic Bertrand curve in the Euclidean space  $E^4$*  for  $r - K = 0$  where  $r$  is the torsion of the spatial quaternionic curve  $\alpha$ ,  $K$  is the principal curvature of the quaternionic curve  $\alpha^{(4)}$ . And then, in the other case, we prove that *there is no quaternionic Bertrand curve in the Euclidean space  $E^4$*  for  $r - K \neq 0$  and  $k \neq 0$ . So, we give an idea of quaternionic Bertrand curve which we call quaternionic  $(N - B_2)$  Bertrand curve in the Euclidean space  $E^4$  by using the similar method which is given by *Matsuda and Yorozi* in [16] and we give some characterizations for quaternionic Bertrand curves and quaternionic  $(N - B_2)$  Bertrand curves. Also, we give some examples of such curves.

## 2. PRELIMINARIES

Let  $Q_H$  denotes a four dimensional vector space over the field  $H$  of characteristic grater than 2. Let  $e_i$  ( $1 \leq i \leq 4$ )denote a basis for the vector space. Let the rule of multiplication on  $Q_H$  be defined on  $e_i$  ( $1 \leq i \leq 4$ ) and extended to the whole of the vector space by distributivity as follows:

A real quaternion is defined with  $q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + de_4$  where  $a, b, c, d$  are ordinary numbers. Such that

$$(2.1) \quad \begin{aligned} e_4 &= 1, & e_1^2 = e_2^2 = e_3^2 &= -1, \\ e_1e_2 &= e_3, & e_2e_3 &= e_1, e_3e_1 &= e_2, \\ e_2e_1 &= -e_3, & e_3e_2 &= -e_1, e_1e_3 &= -e_2. \end{aligned}$$

If we denote  $S_q = d$  and  $\vec{V}_q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$ , we can rewrite real quaternions the basic algebraic form  $q = S_q + \vec{V}_q$  where  $S_q$  is scalar part of  $q$  and  $\vec{V}_q$  is vectorial part. Using these basic products we can now expand the product of two quaternions to give

$$(2.2) \quad p \times q = S_p S_q - \langle \vec{V}_p, \vec{V}_q \rangle + S_p \vec{V}_q + S_q \vec{V}_p + \vec{V}_p \wedge \vec{V}_q \text{ for every } p, q \in Q_H,$$

where we have use the inner and cross products in Euclidean space  $E^3$  [11]. There is a unique involutory antiautomorphism of the quaternion algebra, denoted by the symbol  $\gamma$  and defined as follows:

$$\gamma q = -a\vec{e}_1 - b\vec{e}_2 - c\vec{e}_3 + de_4 \text{ for every } q = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 + de_4 \in Q_H$$

which is called the “Hamiltonian conjugation”. This defines the symmetric, real valued, non-degenerate, bilinear form  $h$  are follows:

$$h(p, q) = \frac{1}{2} [ p \times \gamma q + q \times \gamma p ] \text{ for } p, q \in Q_H.$$

And then, the norm of any  $q$  real quaternion denotes

$$(2.3) \quad \|q\|^2 = h(q, q) = q \times \gamma q.$$

The concept of a spatial quaternion will be used of throughout our work.  $q$  is called a spatial quaternion whenever  $q + \gamma q = 0$ . [1].

The Serret-Frenet formulae for quaternionic curves in  $E^3$  and  $E^4$  are follows:

**Theorem 2.1.** The three-dimensional Euclidean space  $E^3$  is identified with the space of spatial quaternions  $\{p \in Q_H \mid p + \gamma p = 0\}$  in an obvious manner. Let  $I = [0, 1]$  denotes the unit interval of the real line  $\mathbb{R}$ . Let

$$\begin{aligned} \alpha : I \subset \mathbb{R} &\longrightarrow Q_H \\ s &\longrightarrow \alpha(s) = \sum_{i=1}^3 \alpha_i(s) \vec{e}_i, \quad 1 \leq i \leq 3. \end{aligned}$$

be an arc-lenghted curve with nonzero curvatures  $\{k, r\}$  and  $\{t(s), n(s), b(s)\}$  denotes the Frenet frame of the spatial quaternionic curve  $\alpha$ . Then Frenet formulas are given by

$$(2.4) \quad \begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & r \\ 0 & -r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}$$

where  $k$  is the principal curvature,  $r$  is torsion of  $\alpha$  [11].

**Theorem 2.2.** The four-dimensional Euclidean spaces  $E^4$  is identified with the space of quaternions. Let  $I = [0, 1]$  denotes the unit interval of the real line  $\mathbb{R}$ . Let

$$\begin{aligned} \alpha^{(4)} : I \subset \mathbb{R} &\longrightarrow Q_H \\ s \longrightarrow \alpha^{(4)}(s) &= \sum_{i=1}^4 \alpha_i(s) \vec{e}_i, \quad 1 \leq i \leq 4, \quad \vec{e}_4 = 1. \end{aligned}$$

be a smooth curve in  $E^4$  with nonzero curvatures  $\{K, k, r - K\}$  and  $\{T(s), N(s), B_1(s), B_2(s)\}$  denotes the Frenet frame of the quaternionic curve  $\alpha^{(4)}$ . Then Frenet formulas are given by

$$(2.5) \quad \begin{bmatrix} T' \\ N' \\ B'_1 \\ B'_2 \end{bmatrix} = \begin{bmatrix} 0 & K & 0 & 0 \\ -K & 0 & k & 0 \\ 0 & -k & 0 & (r - K) \\ 0 & 0 & -(r - K) & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}$$

where  $K$  is the principal curvature,  $k$  is the torsion and  $(r - K)$  is bitorsion of  $\alpha^{(4)}$  [11].

**Theorem 2.3.** If  $n \geq 4$ , then no  $C^\infty$ -special Frenet curve in  $\mathbb{E}^n$  is a Bertrand curve [16].

**Definition 2.1.** Let  $\alpha(s)$  and  $\beta(s^*)$  be two spatial quaternionic curves in  $\mathbb{E}^3$  with arc-length parameter  $s$  and  $s^*$ , respectively.  $\{t(s), n(s), b(s)\}$  and  $\{t^*(s^*), n^*(s^*), b^*(s^*)\}$  denote Frenet frames of  $\alpha$  and  $\beta$ , respectively. Then  $\alpha$  is called spatial quaternionic Bertrand curve and  $\beta$  is called spatial quaternionic Bertrand mate of the curve  $\alpha$  if  $n(s)$  and  $n^*(s^*)$  are linearly dependent.

**Definition 2.2.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic curves with arc-length parameter  $s$  and  $\bar{s}$ , respectively. Then  $\alpha^{(4)}$  is called quaternionic Bertrand curve and  $\beta^{(4)}$  is called quaternionic Bertrand mate of the curve  $\alpha^{(4)}$  if principal normal vector fields of the curves  $\alpha^{(4)}$  and  $\beta^{(4)}$  are linearly dependent

### 3. CHARACTERIZATIONS OF THE QUATERNIONIC BERTRAND CURVE

In this section, we consider that the bitorsion of the quaternionic Bertrand curve is equal to zero in Euclidean space  $E^4$  and we give some characterizations for the spatial quaternionic Bertrand curve and quaternionic Bertrand curve in Euclidean space  $E^3$  and  $E^4$ , respectively.

**Theorem 3.1.** Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  and  $\beta : I^* \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic curves with arc-length parameter  $s$  and  $s^*$ , respectively. If  $\{\alpha, \beta\}$  is Bertrand curve couple, then corresponding points are a fixed distance apart for all  $s \in I$ .

*Proof.* From Definition (2.1), we can write

$$(3.1) \quad \beta(s) = \alpha(s) + \lambda(s) n(s)$$

Differentiating the (3.1) with respect to  $s$  and by using the Frenet equation, we get

$$\frac{d\beta(s^*)}{ds^*} \frac{ds^*}{ds} = (1 - \lambda(s) k(s)) t(s) + \lambda'(s) n(s) + \lambda(s) r(s) b(s)$$

If we denote  $\frac{d\beta(s^*)}{ds^*} = t^*(s^*)$

$$t^*(s^*) = \frac{ds}{ds^*} [(1 - \lambda(s)k(s))t(s) + \lambda'(s)n(s) + \lambda(s)r(s)b(s)]$$

and

$$h(t^*(s^*), n^*(s^*)) = \frac{ds}{ds^*} \left[ \begin{array}{l} (1 - \lambda(s)k(s))h(t(s), n^*(s^*)) + \lambda'(s)h(n(s), n^*(s^*)) \\ + \lambda(s)r(s)h(b(s), n^*(s^*)) \end{array} \right]$$

Since  $\{n^*(s^*), n(s)\}$  is a linearly dependent set, we have

$$\lambda'(s) = 0$$

that is,  $d(\alpha(s), \beta(s)) = \lambda(s)$  is constant function on  $I$ . This completes the proof.  $\square$

**Theorem 3.2.** Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic Bertrand curve and  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic Bertrand mate of the curve  $\alpha$ . Then, the measure of the angle between the tangent vector fields of spatial quaternionic curves  $\alpha$  and  $\beta$  is constant.

*Proof.* Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  and  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic curves with arc-length  $s$  and  $s^*$ , respectively. Let's consider that

$$(3.2) \quad h(t(s), t^*(s^*)) = \cos \theta$$

Differentiating (3.2) with respect to  $s$ , we get

$$\begin{aligned} \frac{d}{ds} h(t(s), t^*(s^*)) &= h\left(\frac{d}{ds} t(s), t^*(s^*)\right) + h\left(t(s), \frac{dt^*(s^*)}{ds^*} \frac{ds^*}{ds}\right) \\ &= h(k(s)n(s), t^*(s^*)) + h\left(t(s), k^*(s^*)n^*(s^*)\frac{ds^*}{ds}\right) \\ &= 0 \end{aligned}$$

Thus, we have

$$(3.2) \quad h(t(s), t^*(s^*)) = \text{constant}$$

This completes the proof.  $\square$

**Theorem 3.3.** Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  and  $\beta(s) = \alpha(s) + \lambda n(s)$  be spatial quaternionic curves with arc-length parameter  $s$  and  $s^*$ , respectively. Then  $\alpha$  is spatial quaternionic Bertrand curve if and only if

$$\lambda_1 k + \lambda_2 r = 1$$

where  $\lambda_1, \lambda_2$  are constants and  $k$  is the principal curvature,  $r$  is the torsion of the curve  $\alpha$ .

*Proof.* Let  $\alpha$  be spatial quaternionic Bertrand curve. From hypothesis, we have

$$\beta(s) = \alpha(s) + \lambda n(s).$$

Differentiating the above equality and by using the Frenet equations, we get

$$t^*(s^*) = \frac{ds}{ds^*} [(1 - \lambda k(s))t(s) + \lambda r(s)b(s)]$$

as  $\{n^*(s^*), n(s)\}$  is a linearly dependent set, we can write

$$t^*(s^*) = \cos \theta t(s) + \sin \theta b(s)$$

where

$$(3.3) \quad \cos \theta = (1 - \lambda k(s)) \frac{ds}{ds^*}$$

$$(3.4) \quad \sin \theta = \lambda r(s) \frac{ds}{ds^*}$$

From (3.3) and the (3.4), we obtain

$$\frac{\cos \theta}{\sin \theta} = \frac{1 - \lambda k(s)}{\lambda r(s)}$$

If we take  $c = \frac{\cos \theta}{\sin \theta}$ , we get

$$\lambda_1 k + \lambda_2 r = 1$$

where  $\lambda_1 = \lambda$  and  $\lambda_2 = c\lambda$

Conversely, the equation  $\lambda_1 k + \lambda_2 r = 1$  holds. Then it is clearly shown that  $n^*(s^*), n(s)$  are linearly dependent. Which completes the proof.  $\square$

**Theorem 3.4.** Let  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  and  $\beta : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be spatial quaternionic curves with arc-length parameter  $s$  and  $s^*$ , respectively. If  $\{\alpha, \beta\}$  is spatial quaternionic Bertrand curve couple in  $\mathbb{E}^3$ , then the product of the curvature functions  $r(s)$  and  $r^*(s^*)$  at the corresponding points of the curves  $\alpha$  and  $\beta$  is constant.

*Proof.* Let  $\beta$  be spatial quaternionic Bertrand mate of  $\alpha$  curve. Then, we can write

$$\beta(s) = \alpha(s) + \lambda n(s)$$

and from (3.4), we have

$$\sin \theta = \lambda r(s) \frac{ds}{ds^*}$$

Now if we interchange the position of the curves  $\alpha$  and  $\beta$ , we get

$$(3.5) \quad \alpha(s) = \beta(s) - \lambda n^*(s^*)$$

Differentiating (3.5) with respect to  $s$

$$\frac{d\alpha(s)}{ds} = \frac{d\beta(s^*)}{ds^*} \frac{ds^*}{ds} - \lambda \frac{dn^*(s^*)}{ds^*} \frac{ds^*}{ds}$$

and we get

$$(3.6) \quad t(s) = (1 + \lambda k^*) \frac{ds^*}{ds} t^*(s^*) - \lambda r^* \frac{ds^*}{ds} b^*(s^*)$$

and

$$(3.7) \quad t(s) = \cos \theta t^*(s^*) + \sin \theta b^*(s^*)$$

from (3.6) and (3.7), we get

$$(3.8) \quad \cos \theta = (1 + \lambda k^*) \frac{ds^*}{ds}$$

and

$$(3.9) \quad \sin \theta = -\lambda r^* \frac{ds^*}{ds}$$

Multiplying both sides of the equation (3.9) by the corresponding sides of the equation (3.4)

$$\sin^2 \theta = -\lambda^2 r r^*$$

and we obtain

$$rr^* = -\frac{\sin^2 \theta}{\lambda^2} = \text{constant.}$$

Which completes the proof.  $\square$

**Theorem 3.5.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic curves with arc-length parameter  $s$  and  $\bar{s}$ , respectively. If  $\{\alpha^{(4)}, \beta^{(4)}\}$  is Bertrand curve couple, then corresponding points are a fixed distance apart for all  $s \in I$ .

*Proof.* We assume that  $\beta^{(4)}$  is a quaternionic Bertrand mate of  $\alpha^{(4)}$ . Let the pairs of  $\alpha^{(4)}(s)$  and  $\beta^{(4)}(s)$  be corresponding points of  $\alpha^{(4)}$  and  $\beta^{(4)}$ . So, we can write

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda(s)N(s)$$

Differentiating the last equality with respect to  $s$  and by using the Frenet equations, we get

$$\bar{T}(\bar{s}) = \frac{ds}{d\bar{s}} [(1 - \lambda K(s))T(s) + \lambda'(s)N(s) + \lambda(s)k(s)B_1(s)]$$

and

$$h(\bar{T}(\bar{s}), \bar{N}(\bar{s})) = \frac{ds}{d\bar{s}} \left[ (1 - \lambda(s)K(s))h(T(s), \bar{N}(\bar{s})) + \lambda'(s)h(N(s), \bar{N}(\bar{s})) + \lambda(s)k(s)h(B_1(s), \bar{N}(\bar{s})) \right]$$

as  $\{\bar{N}(\bar{s}), N(s)\}$  is a linearly dependent set, we get

$$\lambda'(s) = 0$$

that is,  $d(\alpha^{(4)}(s), \beta^{(4)}(s)) = \lambda(s)$  is constant function on  $I$ . This completes the proof.  $\square$

**Theorem 3.6.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic Bertrand curve and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be quaternionic Bertrand mate of the curve  $\alpha^{(4)}$ . Then, the measure of the angle between the tangent vector fields of quaternionic curves  $\alpha^{(4)}$  and  $\beta^{(4)}$  is constant.

*Proof.* It can be proved same way as the proof of Theorem (3.2)  $\square$

**Theorem 3.7.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a quaternionic curve whose the torsion is non-zero and bitorsion is zero and  $\beta^{(4)}(s) = \alpha(s) + \lambda N(s)$  be spatial quaternionic curves with arc-length parameter  $s$  and  $\bar{s}$ , respectively.. Then  $\alpha^{(4)}$  is a quaternionic Bertrand curve if and only if

$$\lambda K + \mu k = 1$$

where  $\lambda$  and  $\mu$  real constants and  $K$  is the principal curvature,  $k$  is the torsion of the curve  $\alpha^{(4)}$ .

*Proof.* Let  $\beta^{(4)}$  be a quaternionic Bertrand mate of  $\alpha^{(4)}$ . Then we can write

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N(s)$$

Differentiating the last equality considering  $\varphi : I \rightarrow \bar{I}$ ,  $\varphi(s) = \bar{s}$  is a  $C^\infty$  function we get

$$\bar{T}(\varphi(s)) = \frac{ds}{d\bar{s}} [(1 - \lambda K(s))T(s) + \lambda k(s)B_1(s)]$$

If we consider the following equation

$$\bar{T}\varphi(s) = \cos \phi T(s) + \sin \phi B_1(s)$$

we get

$$(3.11) \quad \cos \phi = (1 - \lambda K(s)) \frac{ds}{d\bar{s}},$$

$$(3.12) \quad \sin \phi = \lambda k(s) \frac{ds}{d\bar{s}}.$$

Then taking  $\lambda \frac{\cos \phi}{\sin \phi} = \mu$ , we obtain

$$\lambda K + \mu k = 1$$

Conversely, we assume that  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a quaternionic curve with curvature functions  $K, k \neq 0$  satisfying the relation  $\lambda K + \mu k = 1$  for constant numbers  $\lambda$  and  $\mu$ . Then we can write

$$(3.13) \quad \beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N(s)$$

Differentiating (3.13) with respect to  $s$  and by using the Frenet equation

$$\frac{d\beta^{(4)}(s)}{ds} = (1 - \lambda K(s)) T(s) + \lambda k(s) B_1(s)$$

From the hypothesis, we have  $1 - \lambda K = \mu k$ , thus we get

$$\frac{d\beta^{(4)}(s)}{ds} = k(s) (\mu T(s) + \lambda B_1(s))$$

and considering the equality  $\frac{d\beta^{(4)}(s)}{ds} = \frac{d\beta^{(4)}(\bar{s})}{d\bar{s}} \frac{ds}{d\bar{s}}$ , we have

$$\bar{T}(\bar{s}) = \frac{k}{|k| \sqrt{\lambda^2 + \mu^2}} (\mu T(s) + \lambda B_1(s))$$

If we take  $\frac{1}{\sqrt{\lambda^2 + \mu^2}} = A$ , we can write

$$(3.14) \quad \bar{T}(\bar{s}) = \pm A (\mu T(s) + \lambda B_1(s))$$

Differentiating (3.14) with respect to  $s$  and by using the Frenet equation, we obtain

$$\bar{K}(\bar{s}) \bar{N}(\bar{s}) = \pm A \frac{ds}{d\bar{s}} [\mu K - \lambda k] N(s)$$

where  $\frac{ds}{d\bar{s}} = \frac{1}{|k|} A$ . Thus we have

$$\bar{K}(\bar{s}) \bar{N}(\bar{s}) = \pm \frac{1}{|k|} A^2 [\mu K - \lambda k] N(s)$$

Thus, we obtain  $\bar{N}(\bar{s})$  and  $N(s)$  are linearly dependent. This completes the proof.  $\square$

**Theorem 3.8.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic curves with arc-length parameter  $s$  and  $\bar{s}$  respectively. If  $\{\alpha^{(4)}, \beta^{(4)}\}$  is a quaternionic Bertrand curve couple, then the product of curvature functions  $k(s)$  and  $\bar{k}(\bar{s})$  at the corresponding points of the curves  $\alpha$  and  $\beta$  is constant.

*Proof.* Let  $\beta^{(4)}$  be a quaternionic Bertrand mate of  $\alpha^{(4)}$ . Then we can write

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N(s)$$

If we interchange the position of curves  $\alpha^{(4)}(s)$  and  $\beta^{(4)}(s)$ , we can write

$$(3.15) \quad \alpha^{(4)}(s) = \beta^{(4)}(s) - \lambda \bar{N}(\bar{s})$$

Differentiating (3.15) with respect to  $s$  and by using the Frenet equation, we have

$$T(s) = (1 + \lambda \bar{K}) \bar{T} \frac{d\bar{s}}{ds} - \lambda \bar{k} \bar{B}_1 \frac{d\bar{s}}{ds}$$

where if we consider

$$T(s) = \cos \phi \bar{T}(\bar{s}) + \sin \phi \bar{B}_1(\bar{s})$$

we have

$$(3.16) \quad \cos \phi = (1 + \lambda \bar{K}) \frac{d\bar{s}}{ds},$$

$$(3.17) \quad \sin \phi = -\lambda \bar{k} \frac{d\bar{s}}{ds}.$$

Multiplying both sides of the equation (3.17) by the corresponding sides of the equation (3.12), we obtain

$$k \bar{k} = -\frac{\sin^2 \phi}{\lambda^2} = \text{constant}.$$

This completes the proof.  $\square$

**Theorem 3.9.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic curves with arc-length parameter  $s$  and  $\bar{s}$  respectively. If  $\{\alpha^{(4)}, \beta^{(4)}\}$  is a quaternionic Bertrand curve couple, then there are two constants  $\lambda$  and  $\mu$  such that

$$\mu(k + \bar{k}) + \lambda(K + \bar{K}) = 0$$

where the curvatures  $K, k$  belong to  $\alpha^{(4)}$  and the curvatures  $\bar{K}, \bar{k}$  belong to  $\beta^{(4)}$ .

*Proof.* Assume that  $\{\alpha^{(4)}, \beta^{(4)}\}$  is quaternionic Bertrand curve couple in  $\mathbb{E}^4$ . If we divide the terms of (3.8) by the ones of (3.9), we obtain

$$\frac{\cos \phi}{\sin \phi} = \frac{(1 - \lambda K(s)) \frac{ds}{d\bar{s}}}{\lambda k(s) \frac{ds}{d\bar{s}}} = \text{constant}$$

and if we divide the terms of the equation (3.11) by the ones of the equation (3.12), we have

$$\frac{\cos \phi}{\sin \phi} = -\frac{(1 + \lambda \bar{K}) \frac{d\bar{s}}{ds}}{\lambda \bar{k} \frac{d\bar{s}}{ds}}$$

assuming  $\frac{\cos \phi}{\sin \phi} \lambda = \mu$ , we get

$$(3.18) \quad \mu k + \lambda K = 1$$

and

$$(3.19) \quad \mu \bar{k} + \lambda \bar{K} = -1$$

are obtained. From (3.18) and (3.19)

$$\mu(k + \bar{k}) + \lambda(K + \bar{K}) = 0$$

is obtained. This completes the proof.  $\square$

**Theorem 3.10.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a quaternionic curve whose bi torsion  $r(s) - K(s)$  is equal to zero and  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  be a spatial quaternionic curve associated with  $\alpha^{(4)}$  quaternionic curve. Then  $\alpha$  is spatial quaternionic Bertrand curve if and only if  $\alpha^{(4)}$  is a quaternionic Bertrand curve.

*Proof.* We assume that  $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$  is a spatial quaternionic Bertrand curve and  $\beta$  is a spatial quaternionic Bertrand mate of  $\alpha$  quaternionic curve. Then there are  $\lambda, \mu$  are real constants such that the curvatures of  $\alpha$   $k(s)$  and  $r(s)$  satisfy

$$\lambda k(s) + \mu r(s) = 1$$

where  $\lambda$  is a distance between the corresponding points of quaternionic Bertrand curves  $\alpha$  and  $\beta$ . Since the bitorsion of the quaternionic curve  $\alpha^{(4)}(s)$  vanishes, we can write

$$\mu K(s) + \lambda k(s) = 1$$

Thus,  $\alpha^{(4)}(s)$  is a quaternionic Bertrand curve. It is clearly shown that if  $\alpha^{(4)}$  is quaternionic Bertrand curve, then  $\alpha$  is a spatial quaternionic Bertrand curve.  $\square$

Now, we will confirm the above theorem and for the first time, give an example of quaternionic Bertrand curves  $\alpha^{(4)}$  and  $\beta^{(4)}$  in  $E^4$  and their associated spatial quaternionic curves  $\alpha, \beta$  in  $E^3$

**Example 3.1.** We consider a quaternionic curve  $\alpha^{(4)}$  in  $E^4$  defined by  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$

$$\alpha^{(4)}(s) = \left( \cos \frac{s}{\sqrt{3}}, \sin \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}} \right)$$

for all  $s \in I$ . The curve  $\alpha^{(4)}$  is a regular curve and  $s$  is the arc-length parameter of  $\alpha^{(4)}$  and its curvature functions are as follows

$$K = \frac{1}{3}, \quad k = \frac{\sqrt{2}}{3}, \quad r - K = 0$$

For  $\lambda = 2\sqrt{2}$  and  $\mu = -1$ , the curvatures of  $\alpha^{(4)}$  curve satisfy the relation  $\mu K(s) + \lambda k(s) = 1$ . So,  $\alpha^{(4)}(s)$  is a quaternionic Bertrand curve and we can write its quaternionic Bertrand mate  $\beta^{(4)}$  as follows

$$\beta^{(4)}(s) = \left( 2 \cos \frac{s}{\sqrt{3}}, 2 \sin \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}} \right)$$

Spatial quaternionic curve  $\alpha$  in  $E^3$  associated with quaternionic curve  $\alpha^{(4)}$  in  $E^4$  is given by

$$\alpha(s) = \left( \frac{s}{\sqrt{3}}, \sin \frac{s}{\sqrt{3}} + \cos \frac{s}{\sqrt{3}}, \sin \frac{s}{\sqrt{3}} - \cos \frac{s}{\sqrt{3}} \right)$$

where  $s$  is the arc-length parameter of  $\alpha$  and its curvature functions are as follows

$$k = \frac{\sqrt{2}}{3}, \quad r = \frac{1}{3}$$

For  $\lambda = 2\sqrt{2}$  and  $\mu = -1$ , the curvatures of  $\alpha$  curve satisfy the relation  $\lambda k(s) + \mu r(s) = 1$ . So,  $\alpha(s)$  is a spatial quaternionic Bertrand curve and we can write its spatial quaternionic Bertrand mate  $\beta$  as follows

$$\beta(s) = \left( \frac{s}{\sqrt{3}}, -\sin \frac{s}{\sqrt{3}} - \cos \frac{s}{\sqrt{3}}, -\sin \frac{s}{\sqrt{3}} + \cos \frac{s}{\sqrt{3}} \right)$$

4. CHARACTERIZATIONS OF THE QUATERNIONIC  $(N, B_2)$  BERTRAND CURVE IN  
EUCLIDEAN SPACE  $E^4$

In this section, we consider that the bitorsion of the *quaternionic curve is not equal to zero in Euclidean space  $E^4$* . Then the definition of  $(N, B_2)$  Quaternionic Bertrand curve and some characterizations will be given in Euclidean space  $E^4$ .

Firstly, we prove that there is no quaternionic Bertrand curve in  $\mathbb{E}^4$  if its torsion and bitorsion are not equal to zero.

**Theorem 4.1.** If  $r - K \neq 0$  and  $k(s) \neq 0$ , then no quaternionic curve in  $\mathbb{E}^4$  is a Bertrand curve.

*Proof.* Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be quaternionic Bertrand curve in  $\mathbb{E}^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow \mathbb{E}^4$  be a quaternionic Bertrand mate of  $\alpha^{(4)}$  with arc-length parameter  $s$  and  $\bar{s}$ , respectively. We assume that  $\beta^{(4)}$  is distinct from  $\alpha^{(4)}$ . Let the pairs of  $\alpha^{(4)}(s)$  and  $\beta^{(4)}(\bar{s}) = \beta^{(4)}(\varphi(s))$  (where  $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$  is a regular  $C^\infty$ -function) be of corresponding points of  $\alpha^{(4)}$  and  $\beta^{(4)}$ . Then we can write,

$$(4.1) \quad \beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda(s) N(s)$$

where  $\lambda$  is a  $C^\infty$ -function on  $I$ . Differentiating (4.1) with respect to  $s$  and by using Frenet formulas given in the (2.5), we get

$$(4.2) \quad \varphi'(s) \bar{T}(\varphi(s)) = (1 - \lambda(s) K(s)) T(s) + \lambda'(s) N(s) + \lambda(s) k(s) B_1(s).$$

Since  $h(\bar{T}(\varphi(s)), \bar{N}(\varphi(s))) = 0$  and  $\bar{N}(\varphi(s)) = \pm N(s)$ , we obtain that  $\lambda'(s) = 0$ , that is,  $\lambda(s)$  is a non-zero constant. In this case, we can rewrite equation (4.1) as follows

$$(4.3) \quad \beta^{(4)}(s) = \alpha(s) + \lambda N(s),$$

and we have

$$(4.4) \quad \varphi'(s) \bar{T}(\varphi(s)) = (1 - \lambda K(s)) T(s) + \lambda k(s) B_1(s),$$

By using the equation (4.4), we get,

$$(4.5) \quad \bar{T}(\varphi(s)) = \left( \frac{1 - \lambda(s) K(s)}{\varphi'(s)} \right) T(s) + \left( \frac{\lambda k(s)}{\varphi'(s)} \right) B_1(s).$$

If we denote

$$(4.6) \quad a(s) = \frac{1 - \lambda(s) K(s)}{\varphi'(s)}, \quad b(s) = \frac{\lambda k(s)}{\varphi'(s)}$$

we get,

$$(4.7) \quad \bar{T}(\varphi(s)) = a(s) T(s) + b(s) B_1(s).$$

Differentiating (4.7) with respect to  $s$  and by using Frenet formulas given by (2.5), we have

$$(4.8) \quad \varphi'(s) \bar{K}(\varphi(s)) \bar{N}(\varphi(s)) = a'(s) T(s) + [a(s) K(s) - b(s) k(s)] N(s) + b'(s) B_1(s) + b(s) ((r - K)(s)) B_2(s).$$

Since  $\bar{N}(\varphi(s)) = \pm N(s)$  for all  $s \in I$ , we obtain

$$b(s) ((r - K)(s)) = 0.$$

Since  $(r - K)(s) \neq 0$ , we have  $b(s) = 0$ . From (4.6), we get  $\frac{\lambda k(s)}{\varphi'(s)} = 0$ . Since  $k(s) \neq 0$ , we obtain that  $\lambda = 0$ . Therefore, (4.1) implies that  $\beta^{(4)}$  coincides with  $\alpha^{(4)}$ . This is a contradiction with our assumption. This completes the proof.  $\square$

Now, we give the definition of  $(N - B_2)$  quaternionic Bertrand curves in Euclidean space  $E^4$  and we give some characterizations of such curves. In the first Theorem, we consider the similar method which is given by *Matsuda and Yorozu* in [16]. So the proof of this theorem will be given like in [16].

**Definition 4.1.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow E^4$  be a quaternionic curves and  $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$  is a regular  $C^\infty$ -function such that each point  $\alpha^{(4)}(s)$  of  $\alpha^{(4)}$  corresponds to the point  $\beta^{(4)}(s)$  of  $\beta^{(4)}$  for all  $s \in I$ . If the Frenet  $(N - B_2)$  normal plane at each point  $\alpha^{(4)}(s)$  of  $\alpha^{(4)}$  coincides with the Frenet  $(\bar{N} - \bar{B}_2)$  normal plane at corresponding point  $\beta^{(4)}(s)$  of  $\beta^{(4)}$  for all  $s \in I$  then  $\alpha^{(4)}$  is a quaternionic  $(N - B_2)$  Bertrand curve in  $E^4$  and  $\beta^{(4)}$  is called a quaternionic  $(N - B_2)$  Bertrand mate of  $\alpha^{(4)}$ .

**Theorem 4.2.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$  be a quaternionic curve with curvatures functions  $K, k, r - K$  and  $\alpha : I \subset \mathbb{R} \rightarrow E^3$  be a spatial quaternionic curve associated with  $\alpha^{(4)}$  quaternionic curve in  $E^4$  with the curvatures  $k$  and  $r$ . Then  $\alpha^{(4)}$  is a quaternionic  $(N - B_2)$  Bertrand curve if and only if there exist constant real numbers  $\lambda \neq 0, \mu \neq 0, \gamma, \delta$  satifying

$$(4.9-i) \quad \lambda k(s) - \mu(r - K)(s) \neq 0,$$

$$(4.9-ii) \quad \gamma[\lambda k(s) - \mu(r - K)(s)] + \lambda K(s) = 1,$$

$$(4.9-iii) \quad \gamma K(s) - k(s) = \delta(r - K)(s)$$

$$(4.9-iv) \quad (\gamma^2 - 1)K(s)k(s) + \gamma\{2r(s)K(s) - k^2(s) - r^2(s)\} \neq 0,$$

for all  $s \in I$ .

*Proof.* Let  $\alpha^{(4)}$  be a quaternionic  $(N - B_2)$  Bertrand curve with arc-lenght parameter  $s$  and  $\beta^{(4)}$  be a quaternionic  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$  with arc-lenght parameter  $\bar{s}$ . Then we have

$$(4.10) \quad \beta^{(4)}(\bar{s}) = \beta^{(4)}(\varphi(s)) = \alpha^{(4)}(s) + \lambda(s)N(s) + \mu(s)B_2(s)$$

for all  $s \in I$ , where  $\lambda(s)$  and  $\mu(s)$  are  $C^\infty$ -functions on  $I$ . Differentiating (4.10) with respect to  $s$  and by using the Frenet equations, we have

$$(4.11) \quad \begin{aligned} \bar{T}(\varphi(s))\varphi'(s) &= [1 - \lambda(s)K(s)]T(s) + \lambda'(s)N(s) \\ &\quad + [\lambda(s)k(s) - \mu(s)(r - K)(s)]B_1(s) + \mu'(s)B_2(s) \end{aligned}$$

for all  $s \in I$ .

Since  $\text{Span}\{\bar{N}(\varphi(s)), \bar{B}_2(\varphi(s))\} = \text{Span}\{N(s), B_2(s)\}$  we can write

$$(4.12) \quad \bar{N}(\varphi(s)) = m(s)N(s) + n(s)B_2(s),$$

$$(4.13) \quad \bar{B}_2(\varphi(s)) = p(s)N(s) + q(s)B_2(s)$$

and by using (4.12) and (4.13) then we have

$$\begin{aligned} g(\bar{N}(\varphi(s)), \bar{T}(\varphi(s))\varphi'(s)) &= \lambda'(s)m(s) + \mu'(s)n(s) = 0, \\ g(B_2(\varphi(s)), \bar{T}(\varphi(s))\varphi'(s)) &= \lambda'(s)p(s) + \mu'(s)q(s) = 0, \end{aligned}$$

where  $\begin{vmatrix} m(s) & n(s) \\ p(s) & q(s) \end{vmatrix}$  is non-zero because  $\{\bar{N}(\varphi(s)), \bar{B}_2(\varphi(s))\}$  vector fields must be linear independent. So, we have

$$\lambda'(s) = 0, \mu'(s) = 0$$

that is,  $\lambda$  and  $\mu$  are constant functions on  $I$ .

So, we can rewrite (4.10) and (4.11) for all  $s \in I$ , respectively as

$$(4.14) \quad \beta^{(4)}(\bar{s}) = \beta^{(4)}(\varphi(s)) = \alpha^{(4)}(s) + \lambda N(s) + \mu B_2(s)$$

and

$$(4.15) \quad \bar{T}(\varphi(s)) \varphi'(s) = [1 - \lambda K(s)] T(s) + [\lambda k(s) - \mu(r - K)(s)] B_1(s).$$

where

$$(4.16) \quad (\varphi'(s))^2 = [1 - \lambda K(s)]^2 + [\lambda k(s) - \mu(r - K)(s)]^2 \neq 0$$

for all  $s \in I$ , if we denote

$$(4.17) \quad a(s) = \left[ \frac{1 - \lambda K(s)}{\varphi'(s)} \right], \quad b(s) = \left[ \frac{\lambda k(s) - \mu(r - K)(s)}{\varphi'(s)} \right],$$

it is easy to obtain

$$(4.18) \quad \bar{T}(\varphi(s)) = a(s)T(s) + b(s)B_1(s)$$

where  $a(s)$  and  $b(s)$  are  $C^\infty$ -functions on  $I$ . If we differentiate the last equation (4.18) with respect to  $s$ , we obtain

$$(4.19) \quad \begin{aligned} \bar{K}(\varphi(s)) \bar{N}(\varphi(s)) \varphi'(s) &= a'(s)T(s) + [a(s)K(s) - b(s)k(s)] N(s) \\ &+ b'(s)B_1(s) + b(s)(r - K)(s)B_2(s) \end{aligned}$$

Since  $\bar{N}(\varphi(s)) \in \text{Span}\{N(s), B_2(s)\}$  it holds that

$$a'(s) = 0, b'(s) = 0,$$

that is,  $a$  and  $b$  are constant functions on  $I$ . So, we can rewrite (4.11)

$$(4.20) \quad \bar{K}(\varphi(s)) \bar{N}(\varphi(s)) \varphi'(s) = [aK(s) - bk(s)] N(s) + b(r - K)(s)B_2(s).$$

By using (4.17) we can easily show that

$$(4.21) \quad b(1 - \lambda K(s)) = a(\lambda k(s) - \mu(r - K)(s)),$$

where  $b$  must be a non-zero constant. If we take  $b = 0$ , from (4.19) we get

$$\bar{K}(\varphi(s)) \bar{N}(\varphi(s)) \varphi'(s) = aK(s)N(s),$$

thus we obtain  $\bar{N}(\varphi(s)) = \pm N(s)$  for all  $s \in I$ . This is a contradiction according to the Theorem (4.1). Thus we must consider only the case of  $b \neq 0$ , and then it can easily see that

$$\lambda k(s) - \mu(r - K)(s) \neq 0$$

for all  $s \in I$ . Thus we prove (4.9-i).

If we denote the constant  $\gamma$  by  $\gamma = \frac{a}{b}$  and by using (4.21) we have

$$\gamma(\lambda k(s) - \mu(r - K)(s)) + \lambda K(s) = 1.$$

for all  $s \in I$ . Thus we prove (4.9-ii).

From (4.20), we have

$$\begin{aligned} h(\bar{K}(\varphi(s))\bar{N}(\varphi(s))\varphi'(s), \bar{K}(\varphi(s))\bar{N}(\varphi(s))\varphi'(s)) &= [aK(s) - bk(s)]^2 \\ &\quad + [b(r - K)(s)]^2 \end{aligned}$$

and then

$$(4.22) \quad \{\bar{K}(\varphi(s))\varphi'(s)\}^2 = [aK(s) - bk(s)]^2 + [b(r - K)(s)]^2 \neq 0$$

From (4.17), we obtain

$$\{\bar{K}(\varphi(s))\varphi'(s)\}^2 = (\lambda k(s) - \mu(r - K)(s))^2 \left[ (\gamma K(s) - k(s))^2 + (r - K)^2 \right] (\varphi'(s))^{-2}$$

for all  $s \in I$ . From (4.9-ii) and (4.16) we can easily show that

$$(\varphi'(s))^2 = (\lambda k(s) - \mu(r - K)(s))^2 [\gamma^2 + 1].$$

and then

$$(4.23) \quad \{\bar{K}(\varphi(s))\varphi'(s)\}^2 = \frac{1}{\gamma^2 + 1} \left[ (\gamma K(s) - k(s))^2 + (r - K)^2 \right].$$

Since  $\bar{K}(\varphi(s))\varphi'(s) \neq 0$  by using (4.20) we have

$$(4.24) \quad \bar{N}(\varphi(s)) = m(s)N(s) + n(s)B_2(s)$$

where

$$m(s) = \frac{aK(s) - bk(s)}{\bar{K}(\varphi(s))\varphi'(s)}, \quad n(s) = \frac{b(s)(r - K)}{\bar{K}(\varphi(s))\varphi'(s)}$$

or we can write from (4.17) and (4.9-ii)

$$(4.25) \quad \left. \begin{aligned} m(s) &= \frac{(\lambda k(s) - \mu(r - K)(s))(\gamma K(s) - k(s))}{\bar{K}(\varphi(s))(\varphi'(s))^2} \\ n(s) &= \frac{(\lambda k(s) - \mu(r - K)(s))(r - K)(s)}{\bar{K}(\varphi(s))(\varphi'(s))^2} \end{aligned} \right\}$$

for all  $s \in I$ .

If we differentiate (4.24) with respect to  $s$ , we obtain,

$$(4.26) \quad \begin{aligned} \varphi'(s) \left\{ \begin{aligned} -\bar{K}(\varphi(s))\bar{T}(\varphi(s)) \\ +\bar{k}(\varphi(s))\bar{B}_1(\varphi(s)) \end{aligned} \right\} &= [-K(s)m(s)]T(s) + m'(s)N(s) \\ &\quad + [m(s)k(s) - n(s)(r - K)(s)]B_1(s) \\ &\quad + n'(s)B_2(s), \end{aligned}$$

for all  $s \in I$ . From (4.26), it holds

$$(4.27) \quad m'(s) = 0, \quad n'(s) = 0,$$

that is,  $m(s)$  and  $n(s)$  are constant functions on  $I$ . Then we can easily show that by using (4.25)

$$\frac{m}{n} = \frac{aK(s) - bk(s)}{b(r - K)(s)},$$

if we denote  $\frac{m}{n} = \delta$ , it is obvious that

$$\gamma K(s) - k(s) = \delta(r - K)(s).$$

Thus we prove (4.9-iii).

From (4.15) and (4.26) we have

$$\begin{aligned}\varphi'(s)\bar{k}(\varphi(s))\bar{B}_1(\varphi(s)) &= \bar{K}(\varphi(s))\bar{T}(\varphi(s))\varphi'(s) - K(s)mT(s) \\ &\quad + [mk(s) - n(r - K)(s)]B_1(s) \\ &= \frac{1}{\bar{K}(\varphi(s))(\varphi'(s))^2}\{A(s)T(s) + B(s)B_1(s)\}\end{aligned}$$

or

$$\begin{aligned}\varphi'(s)\bar{k}(\varphi(s))\bar{B}_1(\varphi(s)) &= [\bar{K}(\varphi(s))(1 - \lambda K(s)) - K(s)m]T(s) \\ &\quad + [\bar{K}(\varphi(s))(\lambda k(s) - \mu(r - K)(s)) + mk(s) - n(r - K)(s)]B_1(s)\end{aligned}$$

where

$$(4.28) \quad A(s) = -\frac{(\lambda k(s) - \mu(r - K)(s))}{(\gamma^2 + 1)} \left[ \begin{array}{c} (\gamma^2 - 1)K(s)k(s) \\ +\gamma\left\{K^2(s) - k^2(s) - (r - K)^2(s)\right\} \end{array} \right],$$

and

$$(4.29) \quad B(s) = \frac{\gamma}{\gamma^2 + 1}(\lambda k(s) - \mu(r - K)(s)) \left[ \begin{array}{c} (\gamma^2 - 1)K(s)k(s) \\ +\gamma\left\{K^2(s) - k^2(s) - (r - K)^2(s)\right\} \end{array} \right]$$

Since  $\varphi'(s)\bar{k}(\varphi(s))\bar{B}_1(\varphi(s)) \neq 0$  for  $\forall s \in I$ , we have

$$(\gamma^2 - 1)K(s)k(s) + \gamma\left\{K^2(s) - k^2(s) - (r - K)^2(s)\right\} \neq 0$$

for all  $s \in I$ . Thus we prove (4.9-iv).

Conversely, let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$  be a quaternionic curve with curvatures  $K, k$   $(r - K) \neq 0$  satisfying the equations (4.9-i), (4.9-ii), (4.9-iii), (4.9-iv) for constant numbers  $\lambda, \mu, \delta, \gamma$  and  $\beta^{(4)}$  be a quaternionic curve such as

$$(4.30) \quad \beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N(s) + \mu B_2(s)$$

for all  $s \in I$ . Differentiating (4.30) with respect to  $s$  we have

$$\frac{d\beta^{(4)}(s)}{ds} = (1 - \lambda K(s))T(s) + (\lambda k(s) - \mu(r - K)(s))B_1(s),$$

thus, by using (4.9-ii), we obtain

$$\frac{d\beta^{(4)}(s)}{ds} = (\lambda k(s) - \mu(r - K)(s))(\gamma T(s) + B_1(s))$$

for all  $s \in I$ . Also we get

$$(4.31) \quad \left\| \frac{d\beta^{(4)}(s)}{ds} \right\| = \xi(\lambda k(s) - \mu(r - K)(s))\sqrt{(\gamma^2 + 1)}$$

where  $\xi = \pm 1$ . Then we can write

$$\bar{s} = \varphi(s) = \int_0^s \left\| \frac{d\beta^{(4)}(t)}{dt} \right\| dt \quad (\forall s \in I)$$

where  $\varphi : I \rightarrow \bar{I}$  is a regular  $C^\infty$ -function, and we obtain

$$\varphi'(s) = \xi(\lambda k(s) - \mu(r - K)(s))\sqrt{(\gamma^2 + 1)},$$

for all  $s \in I$ . If we differentiate (4.30) with respect to  $s$ , we have

$$\varphi^l(s) \frac{d\beta^{(4)}(\bar{s})}{d\bar{s}} \Big|_{\bar{s}=\varphi(s)} = (\lambda k(s) - \mu(r - K)(s)) \{\gamma T(s) + B_1(s)\}$$

or

$$(4.32) \quad \bar{T}(\varphi(s)) = \xi(\gamma^2 + 1)^{-\frac{1}{2}} (\gamma T(s) + B_1(s))$$

for all  $s \in I$ . Differentiating (4.32) with respect to  $s$  we have

$$\varphi^l(s) \frac{\bar{T}(\varphi(s))}{d\bar{s}} = \xi(\gamma^2 + 1)^{-\frac{1}{2}} \{(\gamma K(s) - k(s)) N(s) + (r - K)(s) B_2(s)\}$$

and

$$(4.33) \quad \bar{K}(\varphi(s)) = \left\| \frac{\bar{T}(\varphi(s))}{d\bar{s}} \right\| = \frac{\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}}{(\varphi^l(s))\sqrt{(\gamma^2 + 1)}}.$$

for all  $s \in I$ . From the Frenet equations for the curve  $\beta^{(4)}$ , we have  $\frac{\bar{T}(\varphi(s))}{d\bar{s}} = \bar{K}(\varphi(s)) \bar{N}(\varphi(s))$ .

Then we can write

$$\begin{aligned} \bar{N}(\varphi(s)) &= \frac{1}{\bar{K}(\varphi(s))} \frac{d\bar{T}(\varphi(s))}{d\bar{s}} \\ &= \frac{\xi \{(\gamma K(s) - k(s)) N(s) + (r - K)(s) B_2(s)\}}{\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}} \end{aligned}$$

for all  $s \in I$ . If we denote

$$m(s) = \frac{(\gamma K(s) - k(s))}{\xi \sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}}$$

and

$$n(s) = \frac{(r - K)(s)}{\xi \sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}},$$

we have

$$(4.34) \quad \bar{N}(\varphi(s)) = m(s) N(s) + n(s) B_2(s).$$

and we can easily show that  $m(s)$  and  $n(s)$  are constant functions. So differentiating (4.34) with respect to  $s$  and by using the Frenet equations, we have

$$\begin{aligned} \frac{\bar{N}(\varphi(s))}{d\bar{s}} \varphi^l(s) &= m N^l(s) + n B_2^l(s) \\ &= m(K(s)T(s) + k(s)B_1(s)) - n(r - K)(s)B_1(s) \\ &= -mK(s)T(s) + \{mk(s) - n(r - K)(s)\} B_1(s) \end{aligned}$$

or

$$\begin{aligned} (4.35) \quad \frac{\bar{N}(\varphi(s))}{d\bar{s}} &= \frac{-K(s)(\gamma K(s) - k(s))}{\varphi^l(s)\xi\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}} T(s) \\ &\quad + \frac{k(s)(\gamma K(s) - k(s)) - (r - K)^2(s)}{\varphi^l(s)\xi\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}} B_1(s). \end{aligned}$$

for all  $s \in I$ . Also, by using (4.32) and (4.33) we can write

$$(4.36) \quad \begin{aligned} & \bar{K}(\varphi(s))\bar{T}(\varphi(s)) \\ &= \frac{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}{\xi \varphi'(s)(\gamma^2 + 1)\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}} (\gamma T(s) + B_1(s)) \end{aligned}$$

And, then from the Frenet equations for the curve  $\beta$  and (4.36), respectively we have

$$\begin{aligned} \frac{\bar{N}(\varphi(s))}{d\bar{s}} + \bar{K}(\varphi(s))\bar{T}(\varphi(s)) &= \bar{k}(\varphi(s))\bar{B}_1(\varphi(s)), \\ \frac{\bar{N}(\varphi(s))}{d\bar{s}} + \bar{K}(\varphi(s))\bar{T}(\varphi(s)) &= \frac{P(s)}{R(s)}T(s) + \frac{Q(s)}{R(s)}B_1(s) \end{aligned}$$

where we can easily show

$$\begin{aligned} P(s) &= - \left[ \begin{array}{c} (\gamma^2 - 1)K(s)k(s) \\ +\gamma \left\{ K^2(s) - k^2(s) - (r - K)^2(s) \right\} \end{array} \right] \\ Q(s) &= \gamma \left[ \begin{array}{c} (\gamma^2 - 1)K(s)k(s) \\ +\gamma \left\{ K^2(s) - k^2(s) - (r - K)^2(s) \right\} \end{array} \right] \\ R(s) &= \xi \varphi'(s)(\gamma^2 + 1)\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)} \neq 0. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \left\| \frac{\bar{N}(\varphi(s))}{d\bar{s}} + \bar{K}(\varphi(s))\bar{T}(\varphi(s)) \right\| &= \|\bar{k}(\varphi(s))\bar{B}_1(\varphi(s))\| \\ &= \frac{1}{R(s)}\sqrt{P^2(s) + Q^2(s)} \end{aligned}$$

and then

$$\bar{k}(\varphi(s)) = \frac{1}{R(s)}\sqrt{P^2(s) + Q^2(s)}.$$

Moreover, by using Frenet equations for the curve  $\beta^{(4)}$ , we can define unit vector fields  $\bar{B}_1(\varphi(s)), \bar{B}_2(\varphi(s))$ , respectively.

$$(4.37) \quad \bar{B}_1(\varphi(s)) = \frac{1}{\bar{k}(\varphi(s))} \left( \frac{d\bar{N}(\varphi(s))}{d\bar{s}} + \bar{K}(\varphi(s))\bar{T}(\varphi(s)) \right),$$

Also,

$$(4.38) \quad (\bar{r} - \bar{K})(\varphi(s)) = \left\| \frac{d\bar{B}_1(\varphi(s))}{d\bar{s}} + \bar{k}(\varphi(s))\bar{N}(\varphi(s)) \right\|$$

for all  $s \in I$ . By using (4.38),

$$\bar{B}_1(\varphi(s)) = \frac{1}{(\bar{r} - \bar{K})(\varphi(s))} \left( \frac{\bar{B}_1(\varphi(s))}{d\bar{s}} + \bar{k}(\varphi(s))\bar{N}(\varphi(s)) \right)$$

Using the Frenet vectors  $\bar{T}, \bar{N}, \bar{B}_1$  and  $\bar{B}_2$  we can easily see that

$$h(\bar{T}, \bar{T}) = h(\bar{N}, \bar{N}) = h(\bar{B}_1, \bar{B}_1) = h(\bar{B}_2, \bar{B}_2) = 1,$$

and

$$h(\bar{T}, \bar{N}) = h(\bar{T}, \bar{B}_1) = h(\bar{T}, \bar{B}_2) = h(\bar{N}, \bar{B}_1) = h(\bar{N}, \bar{B}_2) = h(\bar{B}_1, \bar{B}_2) = 0,$$

for all  $s \in I$  where  $\{\bar{T}, \bar{N}, \bar{B}_1, \bar{B}_2\}$  is Frenet frame along quaternionic curve  $\beta^4$  in  $E^4$ . And it is fact that  $\text{Span}\{N, B_2\} = \text{Span}\{\bar{N}, \bar{B}_2\}$  where  $(N - B_2)$  normal plane of  $\alpha^{(4)}$  and  $(\bar{N} - \bar{B}_2)$  normal plane of  $\beta^4$ . Consequently,  $\alpha$  is a quaternionic  $(N - B_2)$  Bertrand curve in  $E^4$ . Which completes the proof.  $\square$

**Theorem 4.3.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$  be a quaternionic  $(N - B_2)$  Bertrand curve and  $\beta^{(4)}$  be a quaternionic  $(N - B_2)$  Bertrand mate of  $\alpha^{(4)}$  and  $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$  is a regular  $C^\infty$ -function such that each point  $\alpha^{(4)}(s)$  of  $\alpha^{(4)}$  corresponds to the point  $\beta^{(4)}(\bar{s}) = \beta^{(4)}(\varphi(s))$  of  $\beta^{(4)}$  for all  $s \in I$ . Then the distance between the points  $\alpha^{(4)}(s)$  and  $\beta^{(4)}(\bar{s})$  is constant for all  $s \in I$ .

*Proof.* Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$  be quaternionic  $(N - B_2)$ -Bertrand curve in  $E^4$  and  $\beta^{(4)} : \bar{I} \subset \mathbb{R} \rightarrow E^4$  be a quaternionic  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$ . We assume that  $\beta^{(4)}$  is distinct from  $\alpha^{(4)}$ . Let the pairs of  $\alpha^{(4)}(s)$  and  $\beta^{(4)}(\bar{s}) = \beta^{(4)}(\varphi(s))$  (where  $\varphi : I \rightarrow \bar{I}, \bar{s} = \varphi(s)$  is a regular  $C^\infty$ -function) be of corresponding points of  $\alpha^{(4)}$  and  $\beta^{(4)}$ . Then we can write,

$$\beta^{(4)}(\bar{s}) = \beta^{(4)}(\varphi(s)) = \alpha^{(4)}(s) + \lambda N(s) + \mu B_2(s)$$

where  $\lambda$  and  $\mu$  are non-zero constants. Thus, we can write

$$\beta^{(4)}(\bar{s}) - \alpha^{(4)}(s) = \lambda N(s) + \mu B_2(s)$$

and

$$\|\beta^{(4)}(\bar{s}) - \alpha^{(4)}(s)\| = \sqrt{\lambda^2 + \mu^2}.$$

Since,  $d(\alpha^{(4)}(s), \beta^{(4)}(\bar{s})) = \text{constant}$ . This completes the proof.  $\square$

**Corollary 4.1.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$  be a quaternionic  $(N - B_2)$ -Bertrand curve with curvatures functions  $K(s), k(s), (r - K)(s)$  and  $\beta^{(4)}$  a quaternionic  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$  with curvatures functions  $\bar{K}(\varphi(s)), \bar{k}(\varphi(s)), (\bar{r} - \bar{K})(\varphi(s))$ . Then the relations between these curvatures functions are

$$\bar{K}(\varphi(s)) = \frac{\sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}}{\varphi'(s) \sqrt{(\gamma^2 + 1)}},$$

$$\bar{k}(\varphi(s)) = \frac{|\gamma(K^2(s) - k^2(s) - (r - K)^2) + (\gamma^2 - 1)Kk|}{\varphi'(s) \sqrt{(\gamma^2 + 1)} \sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}},$$

$$(\bar{r} - \bar{K})(\varphi(s)) = \frac{\sqrt{(\gamma^2 + 1)}(r - K)K}{\varphi'(s) \sqrt{(\gamma K(s) - k(s))^2 + (r - K)^2(s)}},$$

*Proof.* It is obvious the proof of Theorem (4.2).  $\square$

Now, we give an example of quaternionic  $(N - B_2)$ -Bertrand curve couple  $\alpha^{(4)}$  and  $\beta^{(4)}$  in  $E^4$  and their associated spatial curves  $\alpha, \beta$  in  $E^3$

**Example 4.1.** Let consider a quaternionic curve  $\alpha^{(4)}$  in  $E^4$  defined by  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow E^4$ ;

$$\alpha^{(4)}(s) = \begin{pmatrix} \cos\left(\frac{2}{\sqrt{5}}s\right) \\ \sin\left(\frac{2}{\sqrt{5}}s\right) \\ \cos\left(\frac{1}{\sqrt{5}}s\right) \\ \sin\left(\frac{1}{\sqrt{5}}s\right) \end{pmatrix}$$

for all  $s \in I$ . The curve  $\alpha^{(4)}$  is a regular curve and  $s$  is the arc-length parameter of  $\alpha^{(4)}$  and its curvature functions are as follows

$$K = \frac{17}{5\sqrt{17}}, \quad k = \frac{6}{5\sqrt{17}}, \quad r - K = \frac{10}{5\sqrt{17}}$$

For  $\lambda = 5\sqrt{17}$ ,  $\mu = -5\sqrt{17}$ ,  $\gamma = -1$ ,  $\delta = -\frac{23}{10}$ , the curvatures of quaternionic curve  $\alpha^{(4)}$  satisfy the relations (4.9-i), (4.9-ii), (4.9-iii), (4.9-iv). Thus  $\alpha^{(4)}$  is a quaternionic  $(N - B_2)$ -Bertrand curve and we can write its quaternionic  $(N - B_2)$ -Bertrand mate curve  $\beta^{(4)}$  as follows:

$$\beta^{(4)}(\bar{s}) = 8 \begin{pmatrix} -3 \cos\left(\frac{2}{16\sqrt{10}}\bar{s}\right) \\ -3 \sin\left(\frac{2}{16\sqrt{10}}\bar{s}\right) \\ 2 \cos\left(\frac{1}{16\sqrt{10}}\bar{s}\right) \\ 2 \sin\left(\frac{1}{16\sqrt{10}}\bar{s}\right) \end{pmatrix}$$

where  $\bar{s} = \varphi(s) = 16\sqrt{2}s$ . Also  $\alpha$  and  $\beta$  spatial quaternionic curves associated with  $\alpha^{(4)}$  and  $\beta^{(4)}$  quaternionic curves in  $E^4$  respectively are given by

$$\alpha(s) = \frac{1}{3\sqrt{17}} \left( \frac{9}{\sqrt{5}}s, 2 \cos\left(\frac{3}{\sqrt{5}}s\right), 2 \sin\left(\frac{3}{\sqrt{5}}s\right) \right)$$

and

$$\beta(\bar{s}) = \frac{16}{\sqrt{37}} \left( \frac{19}{16\sqrt{10}}\bar{s}, -\cos\left(\frac{3}{16\sqrt{10}}\bar{s}\right), -\sin\left(\frac{3}{16\sqrt{10}}\bar{s}\right) \right)$$

The picture of the some projections of the quaternionic  $(N - B_2)$ -Bertrand curve  $\alpha^{(4)}$  and the picture of its associated spatial curves  $\alpha$  are rendered Figure 1.

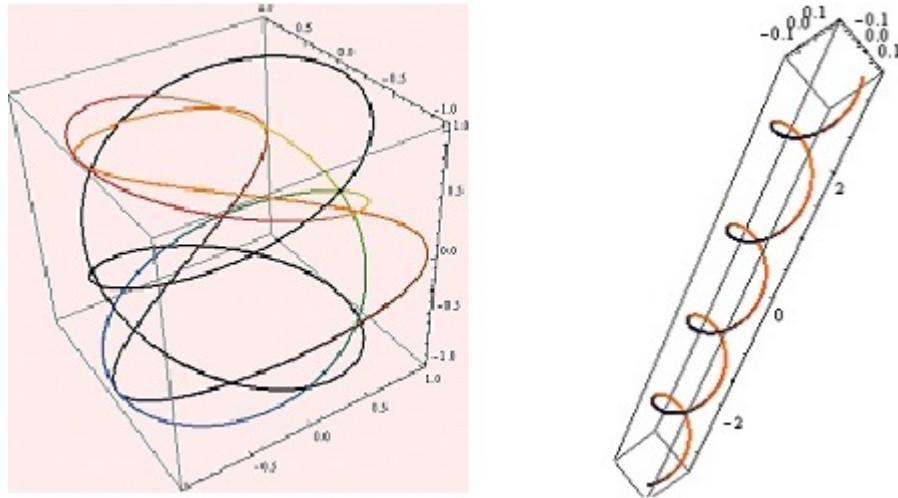


FIGURE 1. Some projection of the quaternionic  $(N - B_2)$ -Bertrand curve  $\alpha^{(4)}$  and the picture of its associated spatial curves  $\alpha$

**Definition 4.2.** Let  $\alpha^{(4)}$  and  $\beta^{(4)}$  be quaternionic curves and we denote Frenet frames of  $\alpha^{(4)}$  and  $\beta^{(4)}$  be  $\{T, N, B_1, B_2\}$  and  $\{\overline{T}, \overline{N}, \overline{B}_1, \overline{B}_2\}$ , respectively. If the angle between the tangent vectors to  $\alpha^{(4)}$  and  $\beta^{(4)}$  at the corresponding points is constant, then the couple  $(\alpha^{(4)}, \beta^{(4)})$  is called quaternionic inclined curve couple.

**Theorem 4.4.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$  be a quaternionic  $(N - B_2)$ -Bertrand curve and  $\beta^{(4)}$  be  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$ . If  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic  $(N - B_2)$ -Bertrand curve couple, then  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic inclined curve couple.

*Proof.* We assume that  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$  be a quaternionic  $(N - B_2)$ -Bertrand curve and  $\beta^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$  be quaternionic  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$ . Then we can write

$$\beta^{(4)}(s) = \alpha^{(4)}(s) + \lambda N(s) + \mu B_2(s)$$

for all  $s \in I$ . From (4.10), we have

$$\overline{T}(\varphi(s)) = a(s)T(s) + b(s)B_1(s)$$

where  $a(s) = \frac{1-\lambda K(s)}{\varphi'(s)}$ ,  $b(s) = \frac{\lambda k(s)-\mu(r-K)(s)}{\varphi'(s)}$ . In this case, we can easily show that  $h(\overline{T}(\varphi(s)), T(s)) = a(s)$ .

Differentiating (4.10) with respect to  $s$  and using the Frenet equations, we obtain

$$\begin{aligned} \overline{K}(\varphi(s)) \overline{N}(\varphi(s)) \varphi'(s) &= a'(s)T(s) + [a(s)K(s) - b(s)k(s)] N(s) \\ &\quad + b'(s)B_1(s) + b(s)(r - K)(s)B_2(s) \end{aligned}$$

Since  $\overline{N}(\varphi(s))$  is expressed by linear combination of  $N(s)$  and  $B_2(s)$ , it holds that

$$a'(s) = 0, b'(s) = 0,$$

that is,  $a$  and  $b$  are constant functions on  $I$ . So,  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic inclined curve couple. This complete the proof.  $\square$

In the following definitions and corollaries were obtained using the notes in [23]

**Definition 4.3.** Let  $\alpha^{(4)}$  and  $\beta^{(4)}$  be quaternionic curves and we denote Frenet frames of  $\alpha^{(4)}$  and  $\beta^{(4)}$  be  $\{T, N, B_1, B_2\}$  and  $\{\overline{T}, \overline{N}, \overline{B}_1, \overline{B}_2\}$ , respectively. If the angle between the normal vectors to  $\alpha^{(4)}$  and  $\beta^{(4)}$  at the corresponding points is constant, then the couple  $(\alpha^{(4)}, \beta^{(4)})$  is called quaternionic slant curve couple.

**Corollary 4.2.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$  be a quaternionic  $(N - B_2)$ -Bertrand curve and  $\beta^{(4)}$  be  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$ . If  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic  $(N - B_2)$ -Bertrand curve couple, then  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic slant curve couple.

*Proof.* It is obvious using the similar method of the above proof.  $\square$

**Definition 4.4.** Let  $\alpha^{(4)}$  and  $\beta^{(4)}$  be quaternionic curves and we denote Frenet frames of  $\alpha^{(4)}$  and  $\beta^{(4)}$  be  $\{T, N, B_1, B_2\}$  and  $\{\overline{T}, \overline{N}, \overline{B}_1, \overline{B}_2\}$ , respectively. If the angle between the first binormal vectors to  $\alpha^{(4)}$  and  $\beta^{(4)}$  at the corresponding points is constant, then the couple  $(\alpha^{(4)}, \beta^{(4)})$  is called quaternionic  $B_1$  slant curve couple.

**Corollary 4.3.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$  be a quaternionic  $(N - B_2)$ -Bertrand curve and  $\beta^{(4)}$  be  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$ . If  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic  $(N - B_2)$ -Bertrand curve couple, then  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic  $B_1$  slant curve couple.

*Proof.* It is obvious using the similar method of the above proof.  $\square$

**Definition 4.5.** Let  $\alpha^{(4)}$  and  $\beta^{(4)}$  be quaternionic curves and we denote Frenet frames of  $\alpha^{(4)}$  and  $\beta^{(4)}$  be  $\{T, N, B_1, B_2\}$  and  $\{\overline{T}, \overline{N}, \overline{B_1}, \overline{B_2}\}$ , respectively. If the angle between the second binormal vectors to  $\alpha^{(4)}$  and  $\beta^{(4)}$  at the corresponding points is constant, then the couple  $(\alpha^{(4)}, \beta^{(4)})$  is called quaternionic  $B_2$  slant curve couple.

**Corollary 4.4.** Let  $\alpha^{(4)} : I \subset \mathbb{R} \rightarrow \mathbb{E}_1^4$  be a quaternionic  $(N - B_2)$ -Bertrand curve and  $\beta^{(4)}$  be  $(N - B_2)$ -Bertrand mate of  $\alpha^{(4)}$ . If  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic  $(N - B_2)$ -Bertrand curve couple, then,  $(\alpha^{(4)}, \beta^{(4)})$  is a quaternionic  $B_2$  slant curve couple.

*Proof.* It is obvious using the similar method of the above proof.  $\square$

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